Dimension Theory in Holomorphic Dynamics

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lecture slides available at
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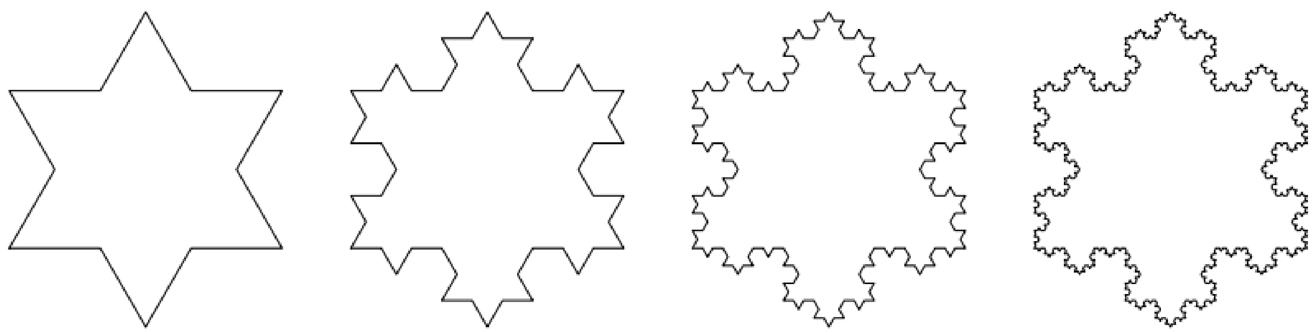
PART I: FRACTAL DIMENSION - 3 WAYS

How do we deduce the complexity of a set K?

Is there some α so that $\#(\text{Boxes to cover } K \text{ of side length } n^{-1}) \simeq n^{\alpha}$?

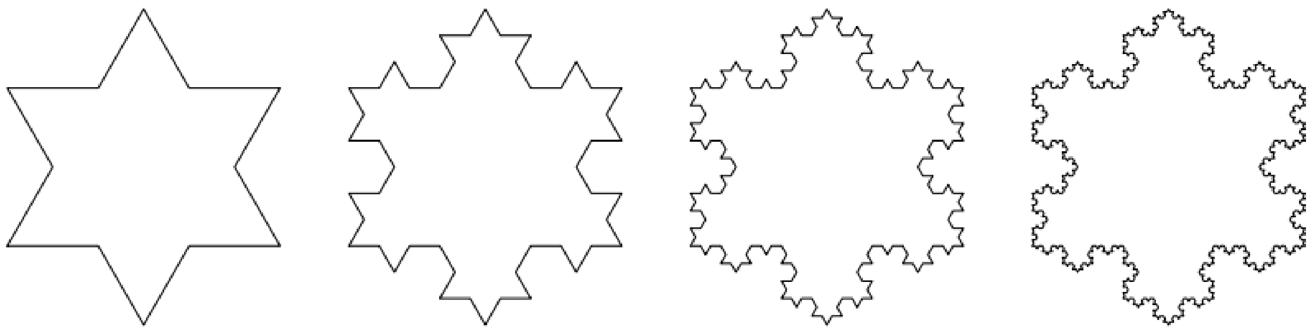
Does that number α correspond to some notion of dimension?

von Koch snowflake - first four generations



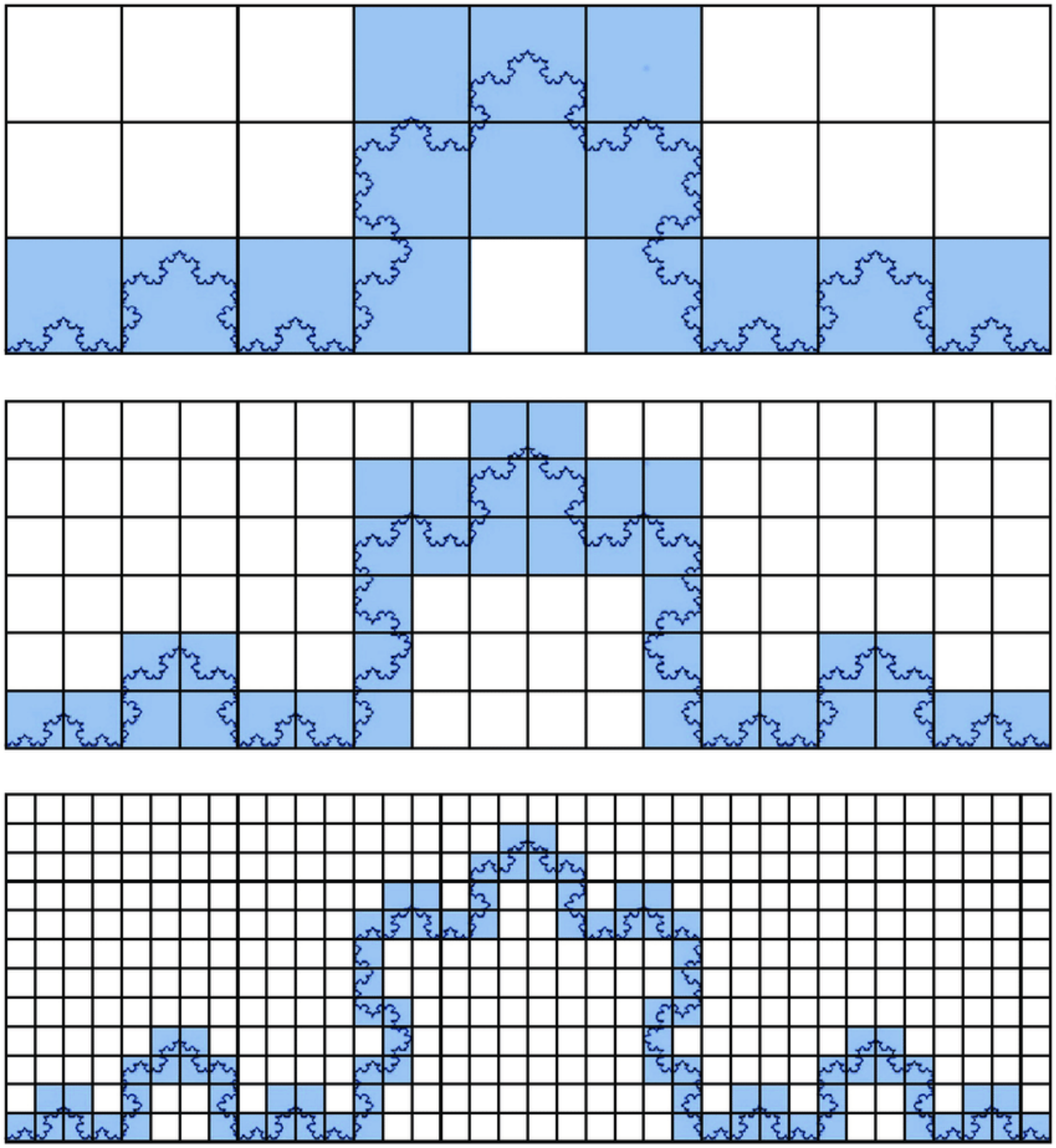
Guesses?

von Koch snowflake - first four generations

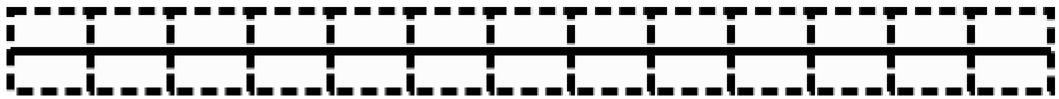


Roughly $n^{\log(4)/\log(3)}$ boxes of side length $1/n$.

von Koch snowflake



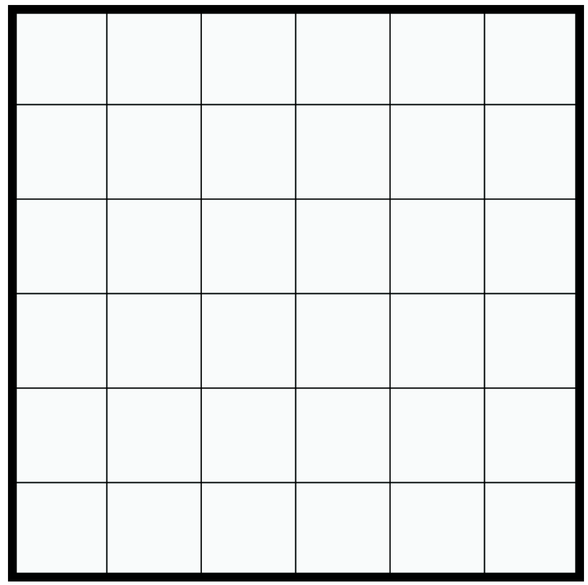
Line segments



n boxes of side length $1/n$.

Growth exponent is 1 - as it should be!

Squares



n^2 boxes of side length $1/n$.

Definition: Let K be a compact set. Let $N(K, \epsilon)$ denote the minimal amount of squares of side length ϵ needed to cover K.

The **upper Minkowski dimension** of K is

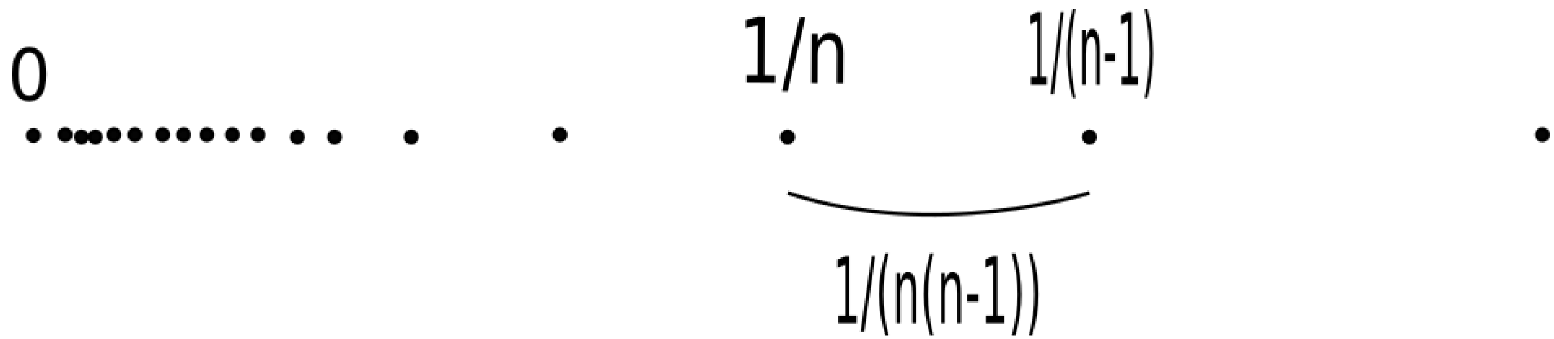
$$\overline{\dim_M}(K) = \limsup_{\epsilon \to 0} \frac{\log(N(K, \epsilon))}{-\log(\epsilon)}.$$

The **lower Minkowski dimension** of K is

$$\underline{\dim_M}(K) = \liminf_{\epsilon \to 0} \frac{\log(N(K, \epsilon))}{-\log(\epsilon)}.$$

If the limit exists, then the **Minkowski dimension** $\dim_M(K)$ is well-defined.

A bad example. Let $K = \{1/n\}_{n=1}^{\infty} \cup \{0\}$.



Countable set, but $\dim_M(K) = 1/2$!

$$\dim_M(\cup K_n) \neq \sup \dim_M(K_n)$$

Countable sets “should” have dimension 0.

One issue - must cover by squares of same/comparable diameter.

What if we drop this condition?

Definition: Let $\alpha > 0$. The α**-Hausdorff content** of a set K is

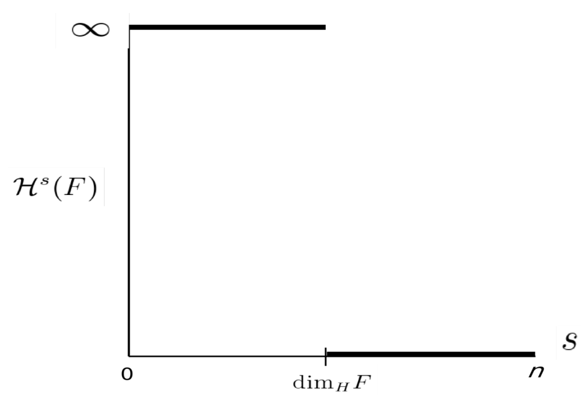
$$H^\alpha_\infty(K) = \inf\left\{\sum_{n=1}^{\infty}\operatorname{diam}(U_n)^\alpha \;:\; K \subset \left(\bigcup_{n=1}^{\infty} U_n\right)\right\}.$$

Infimum taken over all countable covers by open sets $\{U_n\}$

Easy exercise: $H^\alpha_\infty\left(\{0\}\cup\{1/n\}_{n=1}^\infty\right) = 0$ for all $\alpha > 0$.

Definition: The **Hausdorff dimension** of a set K is

$$\dim_H(K) = \sup\{\alpha : H^\alpha(K) > 0\}.$$



In general, for a compact set K we have $\dim_H(K) \leq \dim_M(K)$.

$\dim_H \left(\{0\} \cup \{1/n\}_{n=1}^{\infty}\right) = 0$. Inequality can be strict.

Easy exercise: $\dim_H(\cup K_n) = \sup \dim_H(K)$.

How else can we "fix" Minkowski dimension?

Definition: Let K be a set. Then the **packing dimension** of K is

$$\dim_P(K) = \inf_{\text{covers}} \sup \left\{ \overline{\dim_M}(K_n) \,:\, K \subset \left(\bigcup_{n=1}^{\infty} K_n \right) \right\}.$$

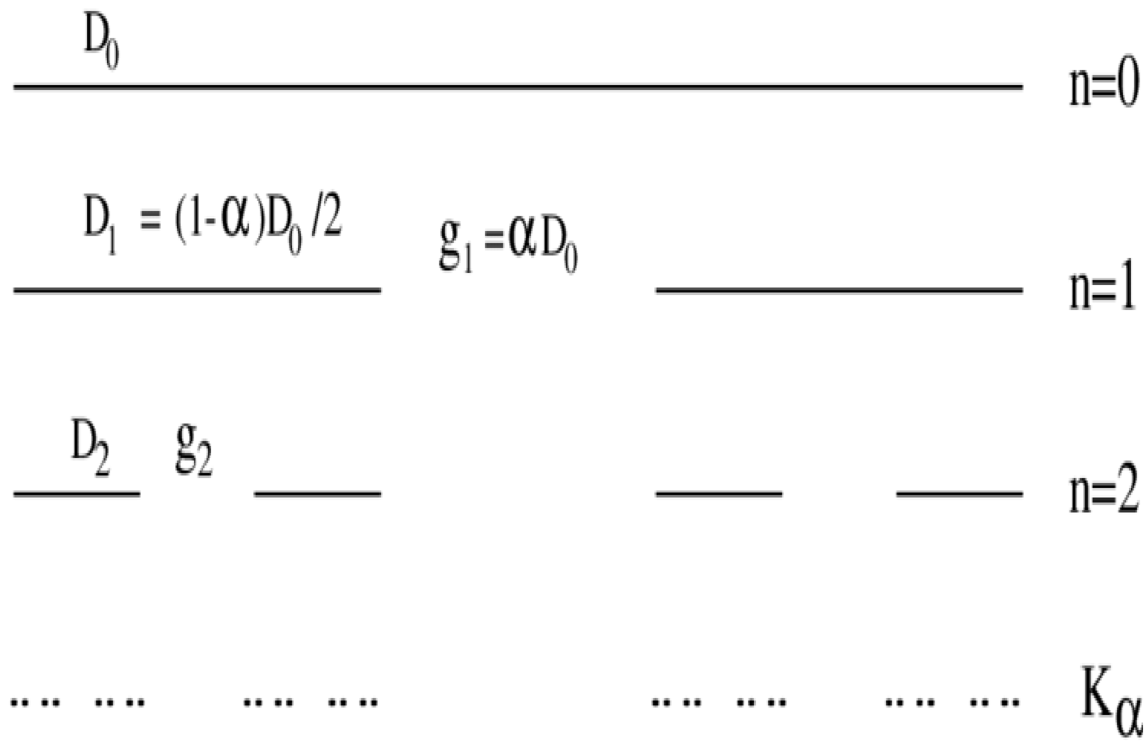
We have modified Minkowski to automatically satisfy

$$\dim_P(\cup K_i) = \sup \dim_P(K)$$

For a given compact set K:

$$\dim_H(K) \leq \dim_P(K) \leq \dim_M(K).$$

When do packing and Hausdorff dimension disagree?



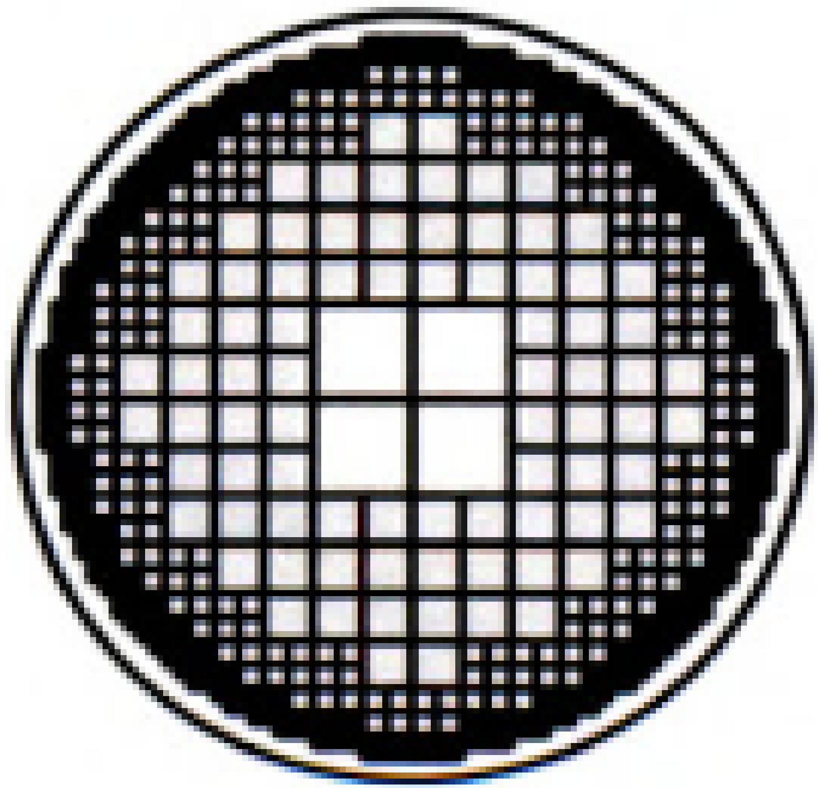
Packing dimension sees the "big" part of a set at all scales.

Hausdorff dimension sees the "small" part of the set at all scales.

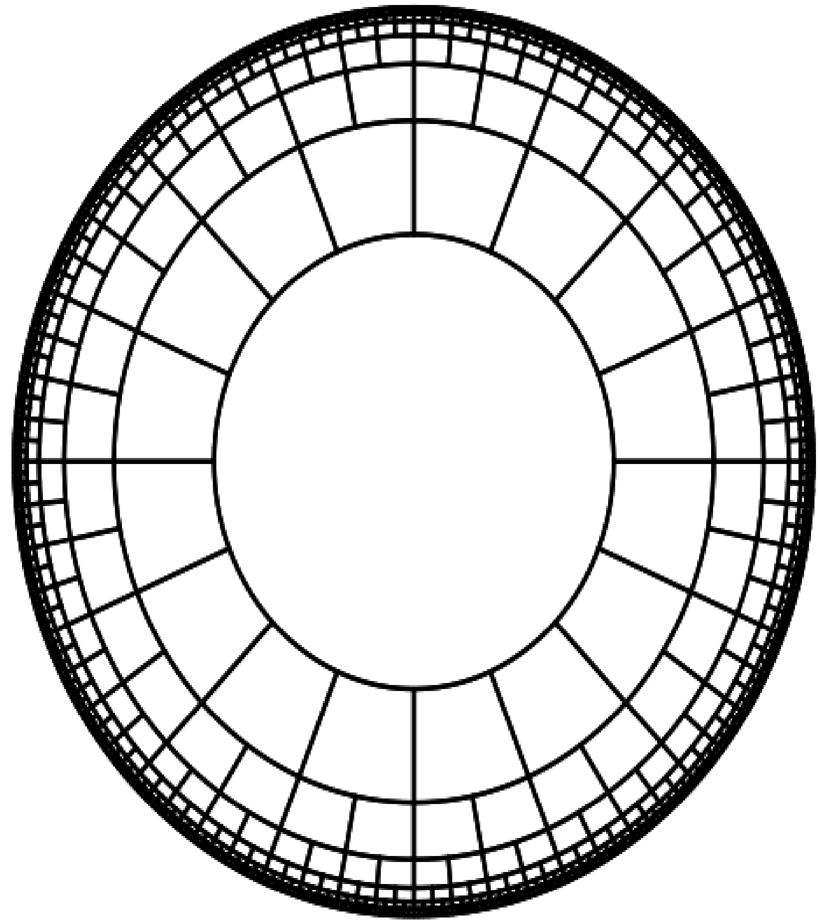
Definition: A **Whitney decomposition** of a bounded open set Ω into squares is a collection of open squares $\{Q_j\}$ satisfying:

1. The cubes have pairwise disjoint interior.
2. $\Omega = \cup \overline{Q_j}$.
3. There exists a constant C so that
$$\frac{1}{C}\text{dist}(Q_j, \partial\Omega) \leq \text{diam}(Q_j) \leq C\text{dist}(Q_j, \partial\Omega)$$

The collection $\{Q_j\}$ need not be literal cubes, so long as the boundaries of the Q_j have zero measure.



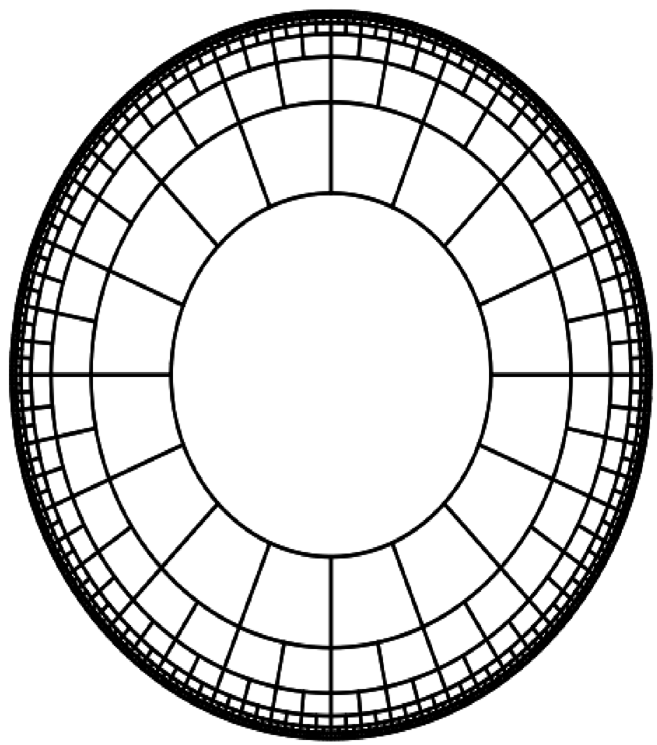
Whitney decomposition of $\mathbb{D}$ with dyadic squares.



Whitney decomposition of $\mathbb{D}$ with hyperbolic squares.

Definition: The **critical exponent** of a Whitney decomposition of the complement of a compact set K is

$$\alpha(K) = \inf \left\{ \alpha : \sum \operatorname{diam}(Q)^{\alpha} < \infty \right\}$$



Example: $\sum \operatorname{diam}(Q)^t \asymp \frac{1}{t-1} \operatorname{diam}(\mathbb{D})^t$

Upper Minkowski dimension and critical exponents are related as follows:

Theorem: Let K be a compact set with zero Lebesgue measure. Then

$$\overline{\dim}_M(K) = \alpha(K).$$

Number of small squares surrounding a set K is related to number of small squares to cover a set.

PART II: HOLOMORPHIC DYNAMICS

Definition: Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.

1. The n**th iterate** of f is $f^{\circ n} := f^n$.
2. The **orbit** of z is the sequence $\{f^n(z)\}$.
3. If f is not a polynomial, f is called **transcendental entire**, or t.e.f.

Theorem (Picard): If f is a t.e.f, then with at most one **exceptional point**, $f^{-1}(\{z\})$ is infinite!

Polynomials much simpler - branched coverings, extend to $\hat{\mathbb{C}}$.

Definition: Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.

The **Fatou set**, $\mathcal{F}(f)$, is the set of all points z such that there exists a ball $B = B(z, r)$ so that $\{f^n|_B\}$ is a normal family.

Normal family $\simeq$ equicontinuity of the family $\{f^n\}$.

Fatou set $\simeq$ "Stable" set for dynamics of f.

Definition: Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function.

The **Julia set**, $\mathcal{J}(f)$, is the complement of the Fatou set in $\mathbb{C}$.

Locally no equicontinuity $\simeq$ nearby points have different orbits!

Julia set $\simeq$ "Chaotic" set for dynamics. Closed set with fractal structure.

Very Simple Example: $f(z) = z^2$.

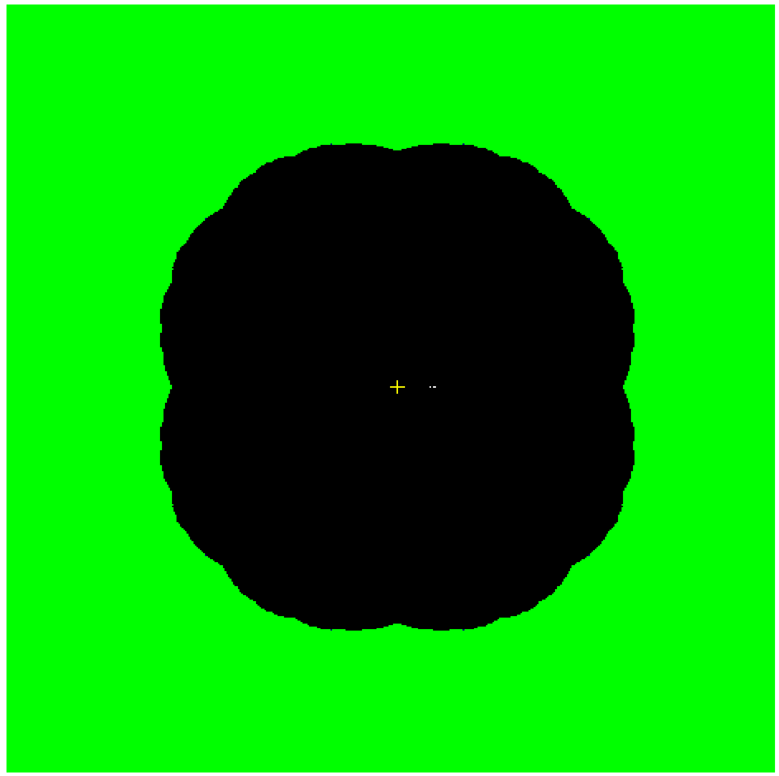
If $|z| < 1$, $f^n(z)$ converges locally uniformly to the constant 0 function - Fatou set!

If $|z| > 1$, $f^n(z)$ converges locally uniformly to ∞ - Fatou set!

If $|z| = 1$, z is near points w with $|w| < 1$ and $|w| > 1$ - Julia set the circle! (Dimension 1).

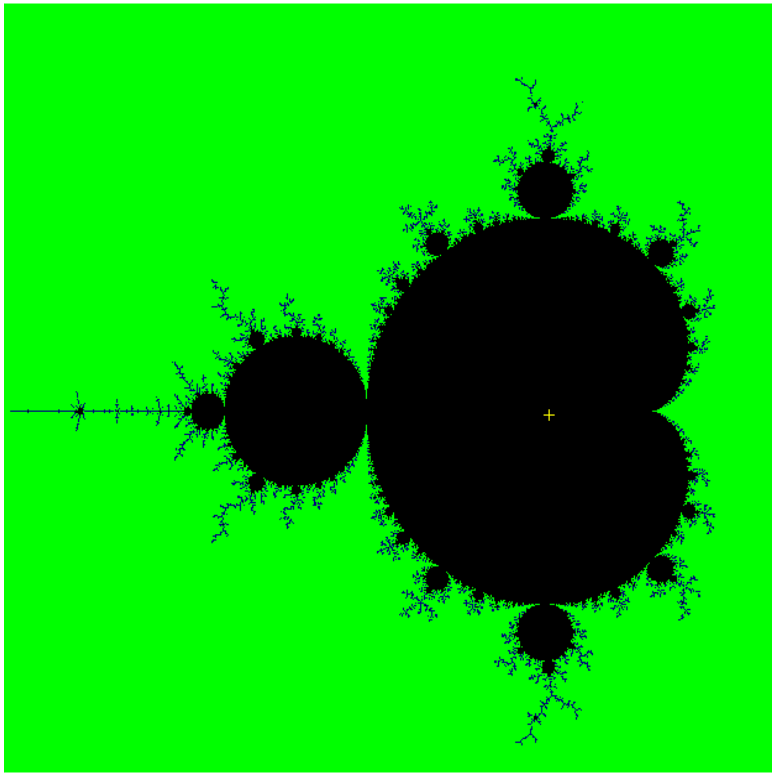
The unit disk $\mathbb{D}$ is an **attracting basin**.

What happens if we add a small c? $f_c(z) := z^2 + c$. Try $c = 1/8$.



Critical point 0 belongs to attracting basin - **hyperbolicity**

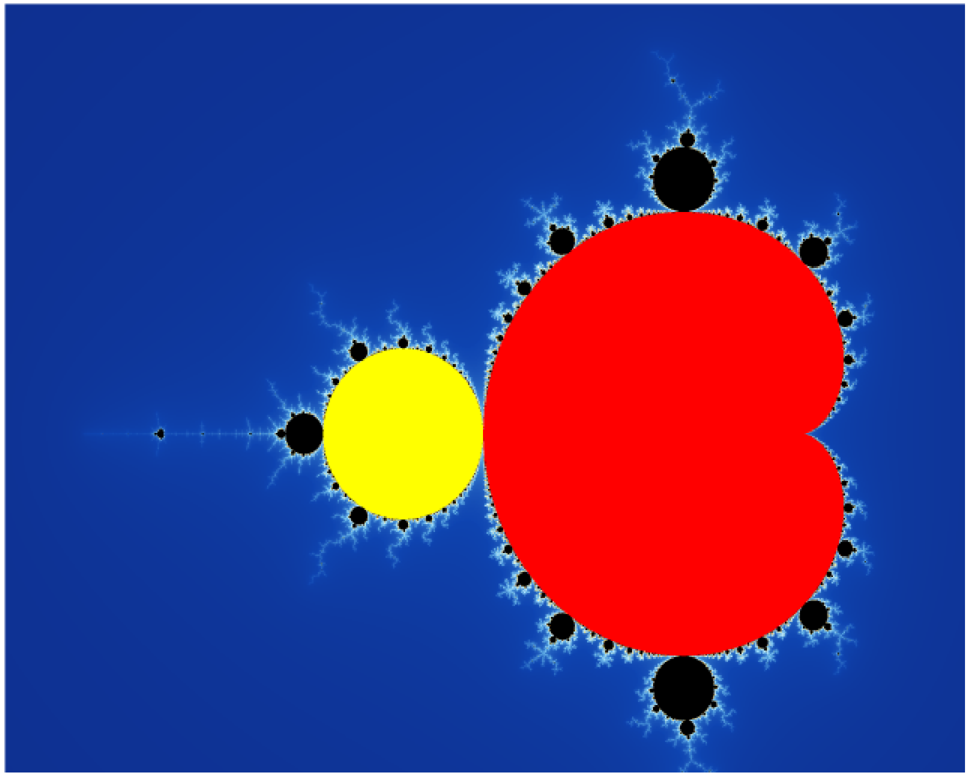
Mandelbrot Set: parameter plane for $f_c(z) = z^2 + c$



$$M = \{c : f_c^n(0) \text{ is bounded}\} = \{c : \mathcal{J}(f_c) \text{ is connected}\}$$

Fractal structure of boundary = notorious open problems.

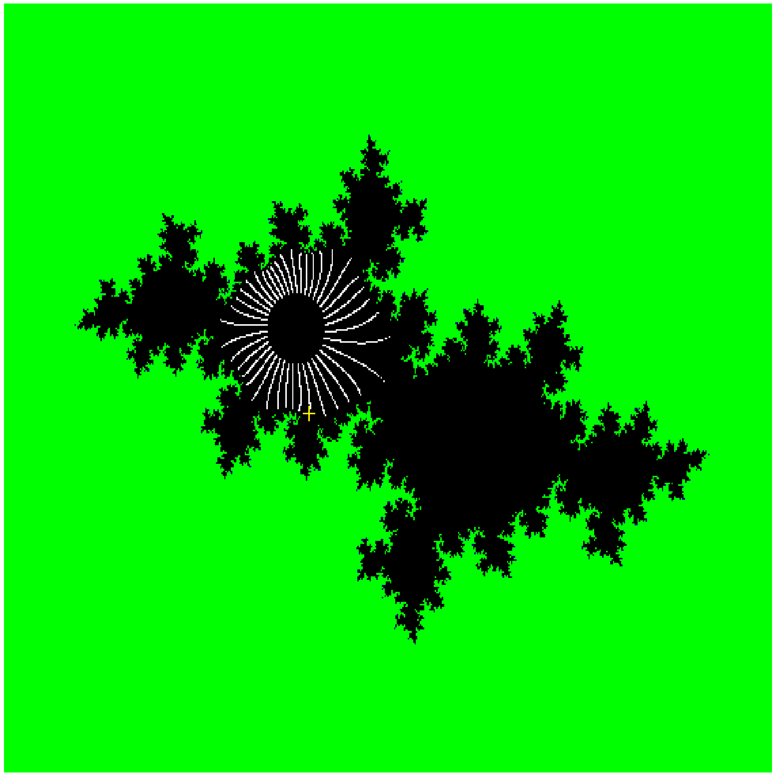
Main Cardioid



Julia sets in the main cardioid are quasicircles.

The Fatou set is a single attracting basin - similar to $z^2 + 1/8$ before.

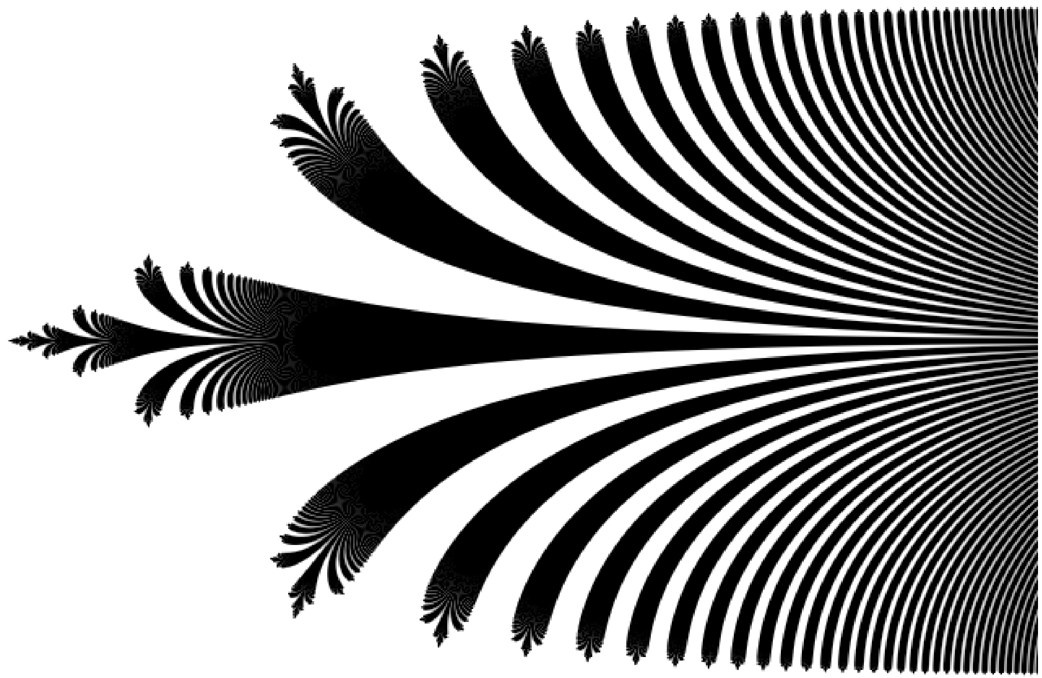
What happens close to boundary of the main cardioid?



$c = -0.592280185953905 + i0.429132211809624$

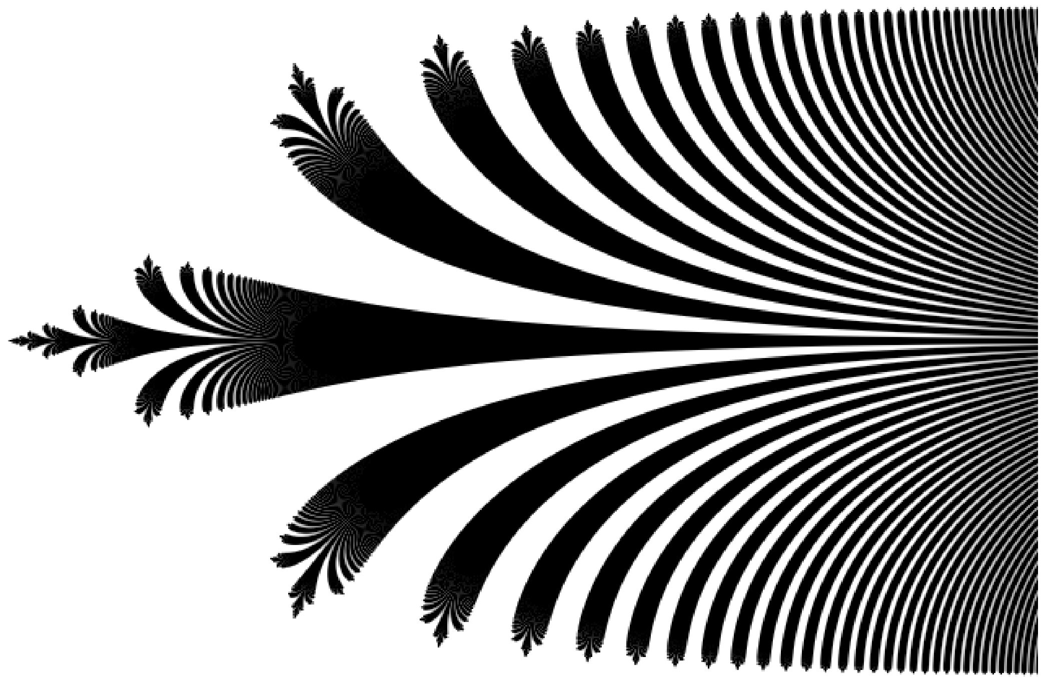
Still an attracting basin!

Julia set of $f(z) = (\exp(z) - 1)/2$.



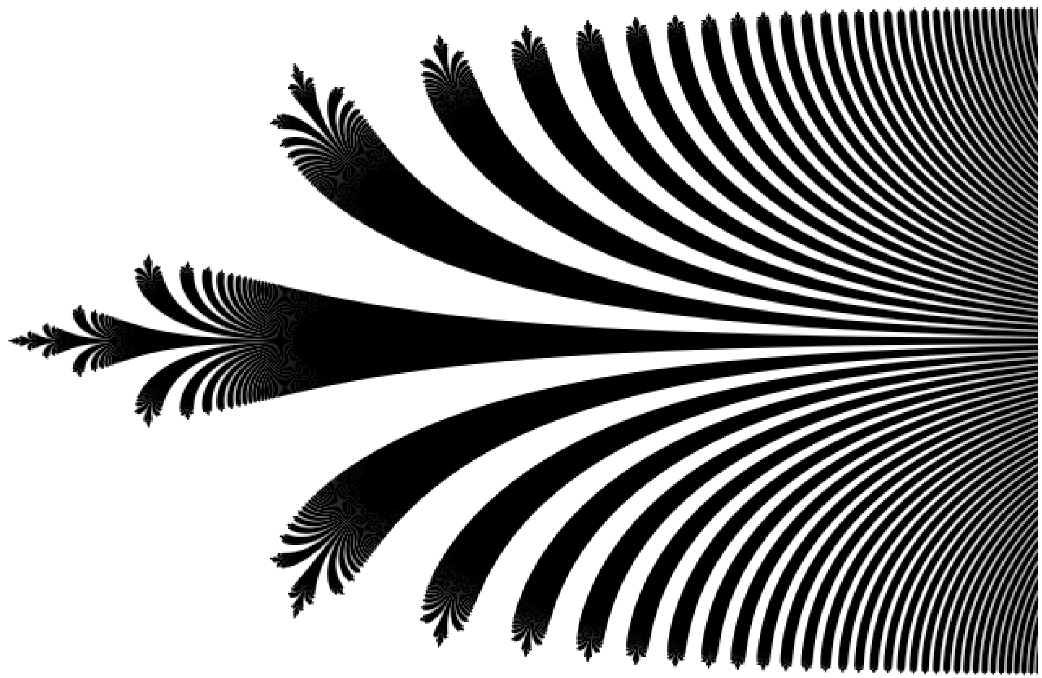
Julia set is a **Cantor bouquet**. Uncountably many rays out of ∞.

Julia set of $f(z) = (\exp(z) - 1)/2$.



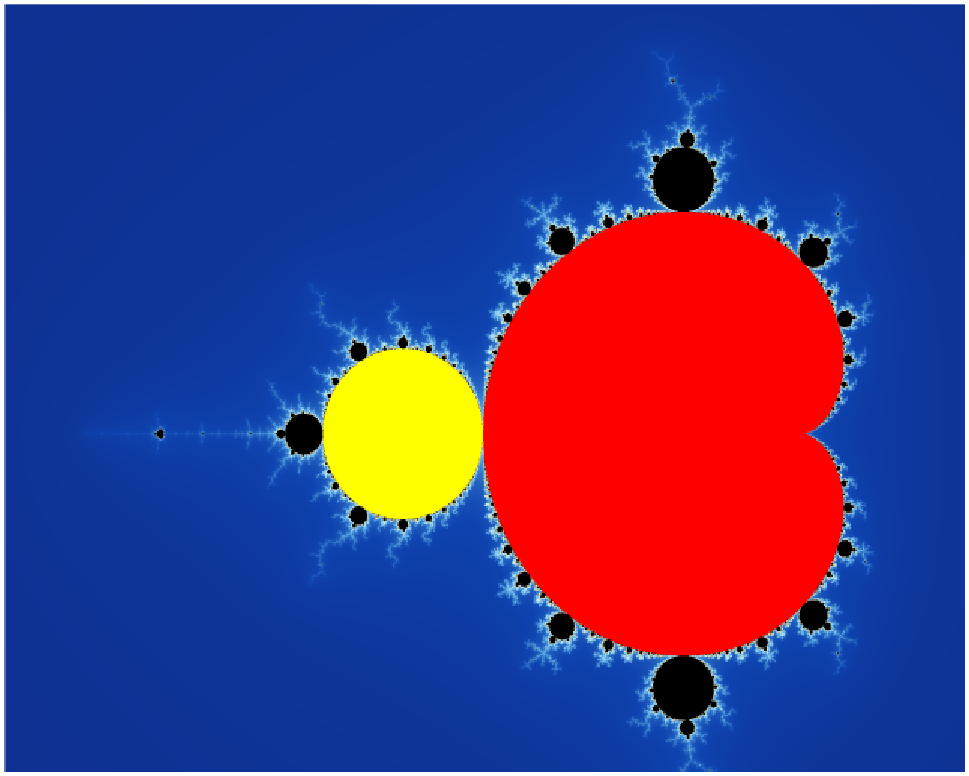
$\dim_H(\mathcal{J}(f)) = 2$, but $\dim_H(\mathcal{J}(f) \setminus \{\text{endpoints of rays}\}) = 1$!

Julia set of $f(z) = (\exp(z) - 1)/2$.

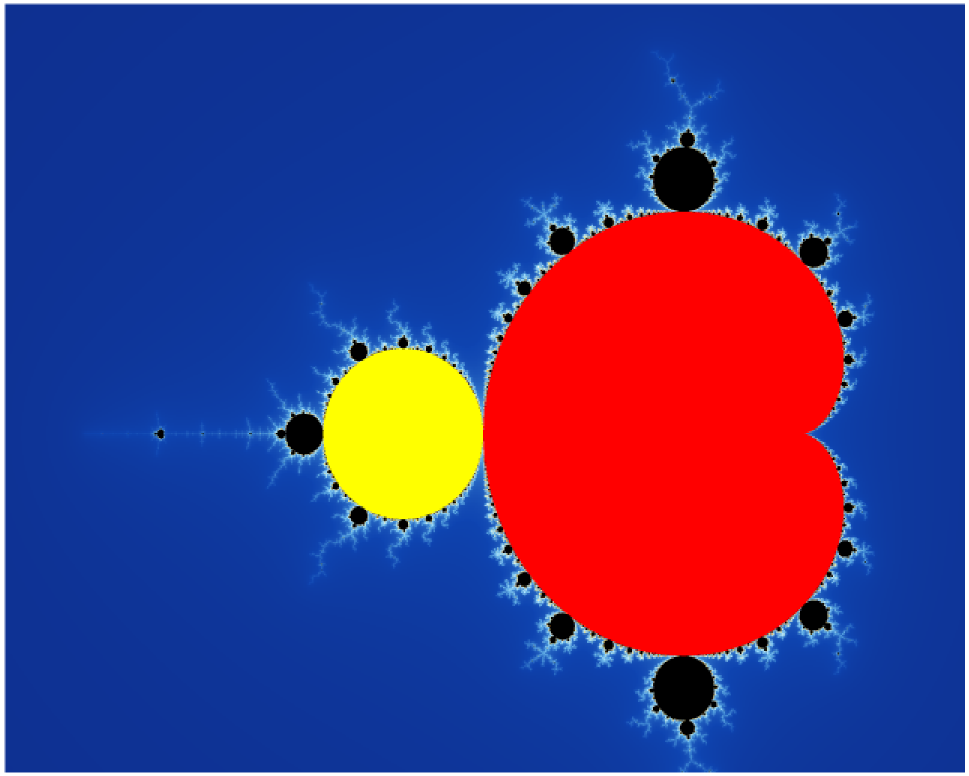


$f \in \mathcal{B}$, Eremenko-Lyubich class. Some similar theory to polynomials.

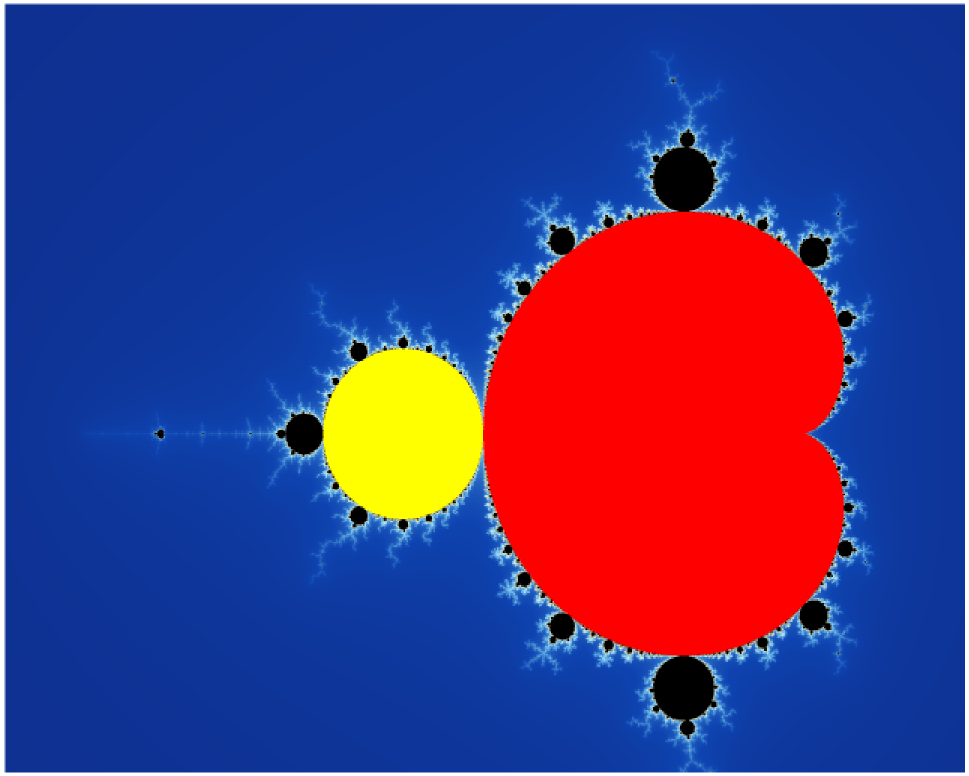
PART III: DIMENSION IN HOLOMORPHIC DYNAMICS



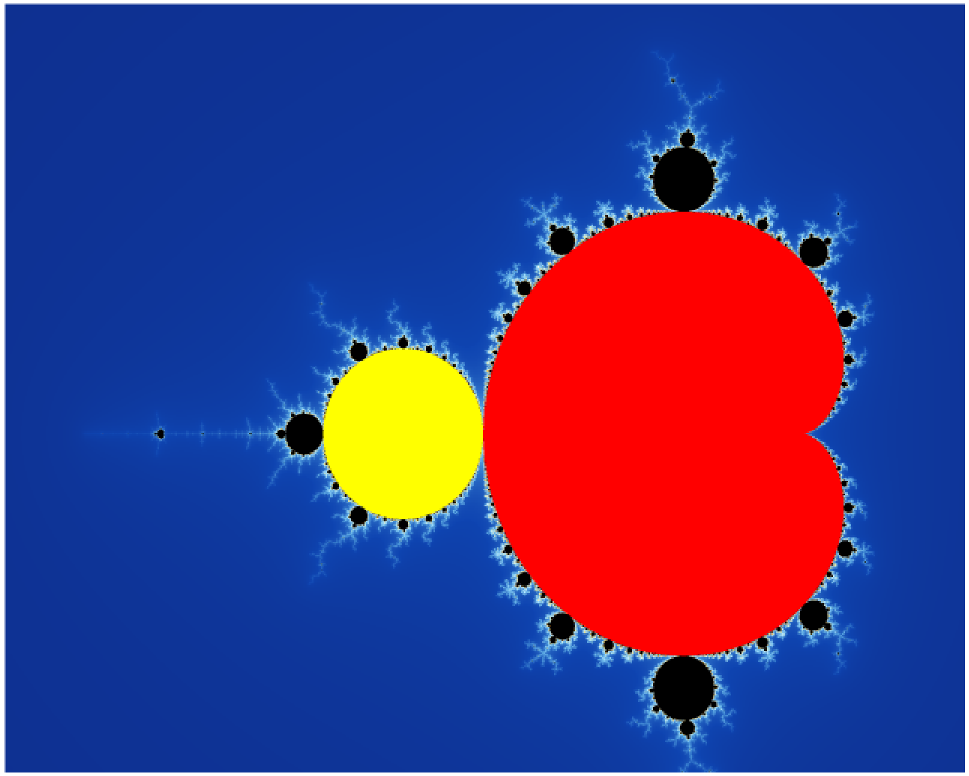
Theorem (Shishikura): The boundary of the Mandelbrot set has Hausdorff dimension 2.



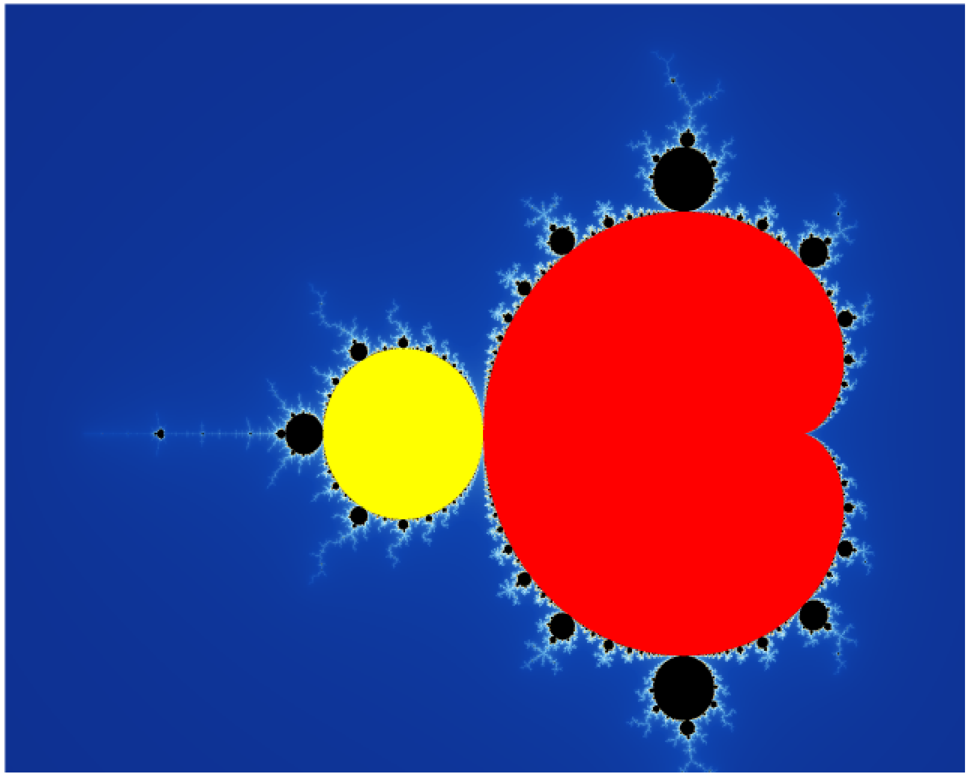
Theorem (Shishikura): The supremum of $\dim_H(\mathcal{J}(f_c))$, c in the main cardioid, is 2.



Theorem (Shishikura): There exists c in the boundary of the main cardioid so that $\dim_H(\mathcal{J}(f_c)) = 2$.



Theorem (Ruelle): The function $c \mapsto \dim_H(\mathcal{J}(f_c))$ is real analytic in the main cardioid.



Theorem (Sullivan): Special measure on hyperbolic Julia sets.

$$\dim_H(\mathcal{J}(z^2+c)) = \dim_P(\mathcal{J}(z^2+c)) = \dim_M(\mathcal{J}(z^2+c)) = t.$$

Theorem (Buff & Cheritat): Quadratic family has positive area Julia sets!

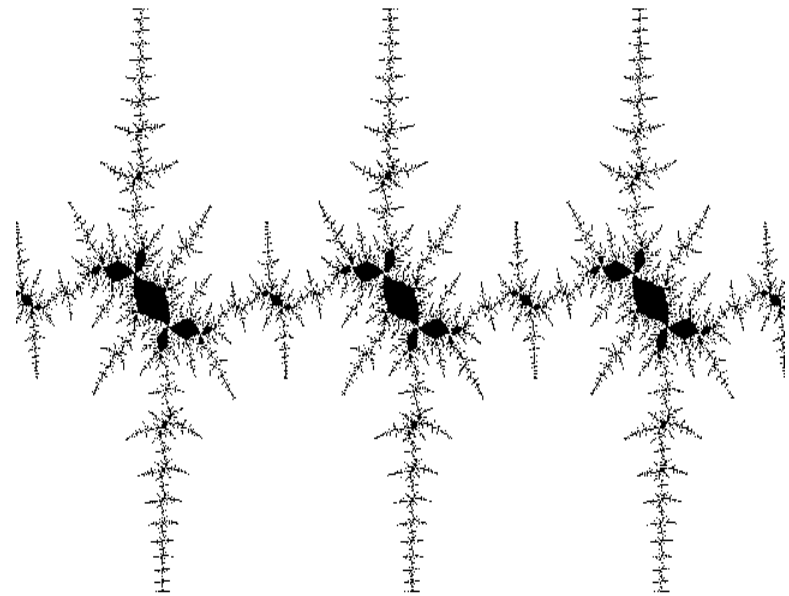
In polynomial dynamics, it is easy to construct examples with Julia sets with small dimensions, but difficult to approach dimension 2 and positive area.

In transcendental dynamics, the problem is the opposite!

Theorem (Baker): Julia sets of t.e.f.s contain non-degenerate continua. Hausdorff dimension lower bounded by 1.

Theorem (Misiurewicz): Julia set of $\exp(z) = \mathbb{C}$.

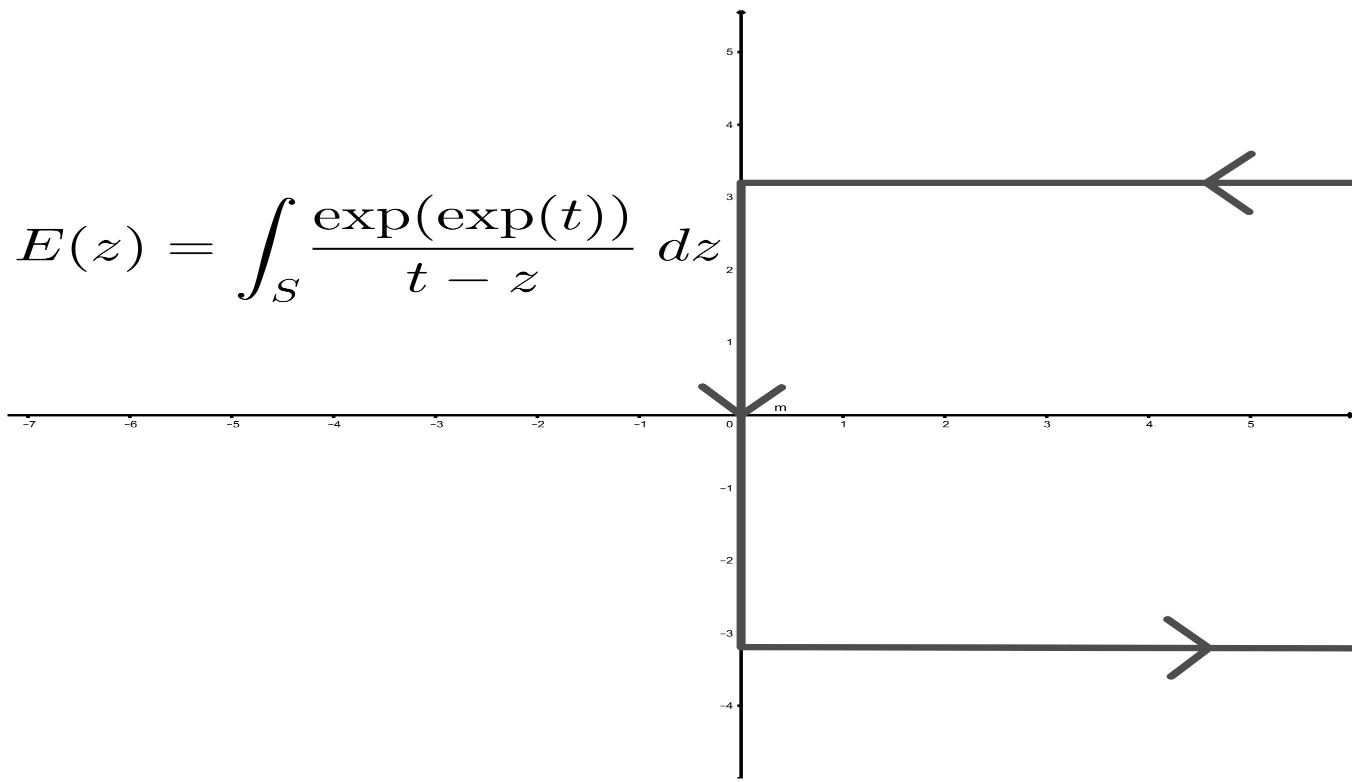
Theorem (McMullen):
$\sin(az+b)$ family always has positive area. $\lambda \exp(z)$ family always has dimension 2. Zero area if there is an attracting cycle.



Julia set in the cosine family.

Theorem (Stallard): There exist functions in $\mathcal{B}$ with Julia set with dimension arbitrarily close to 1; dimension 1 does not occur in $\mathcal{B}$. All dimensions in $(1,2]$ occur in $\mathcal{B}$.

$$E(z) = \int_S \frac{\exp(\exp(t))}{t-z}\, dz$$

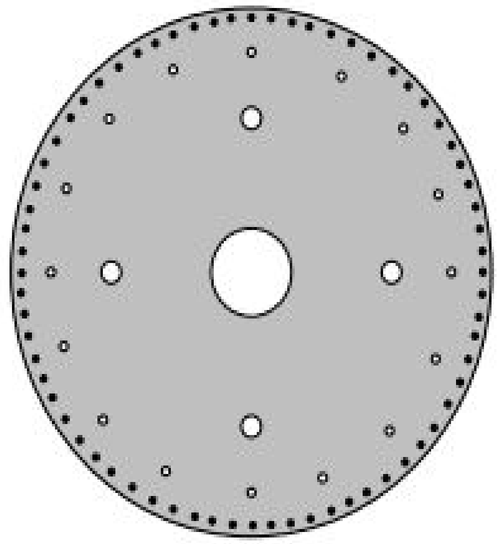


$E_K(z) = E(z) - K$. Dimension tends to 1 as K increases

Theorem (Rippon, Stallard): If $f \in \mathcal{B}$, $\dim_P(J(f)) = 2$.

Compare main cardioid results with results for functions in $\mathcal{B}$.

Theorem (Bishop): There exists a transcendental entire function whose Julia set has Hausdorff dimension AND packing dimension equal to 1.



The functions are of the form

$$f_{\lambda,R,N}(z) = [\lambda(2z^2 - 1)]^{\circ N} \cdot \prod_{k=1}^{\infty} \left(1 - \frac{1}{2}\left(\frac{z}{R_k}\right)^{n_k}\right).$$

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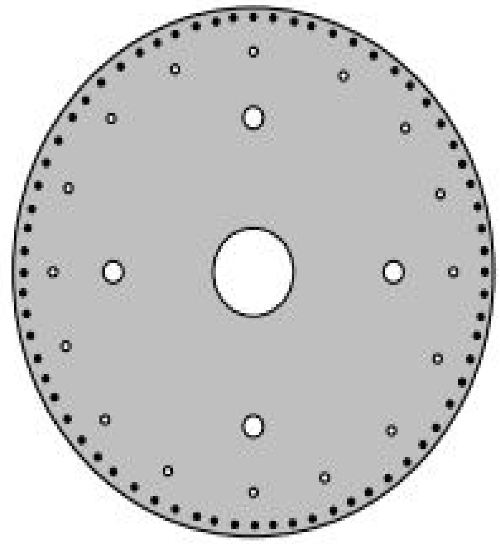

The Julia set looks like the following:

1. A Cantor set near the origin with very small dimension.

2. Boundaries of Fatou components are C^1 "almost"-circles.

3. "Buried" points with very small dimension.

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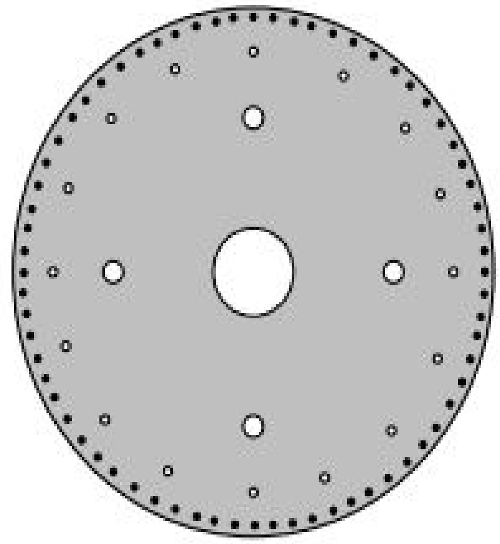

The Julia set looks like the following:

1. A Cantor set near the origin with very small dimension.
2. **Boundaries of Fatou components are C^1 "almost"-circles.**
3. "Buried" points with very small dimension.

The dimension lives on the C^1 almost-circles. Dynamics here are simple.

Theorem (B.): There exists transcendental entire functions with packing dimension in $(1, 2)$.

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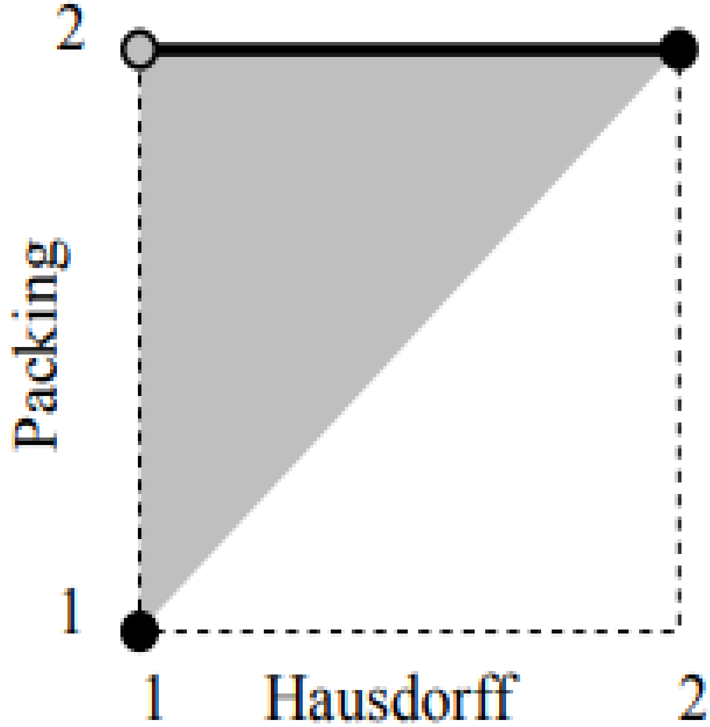
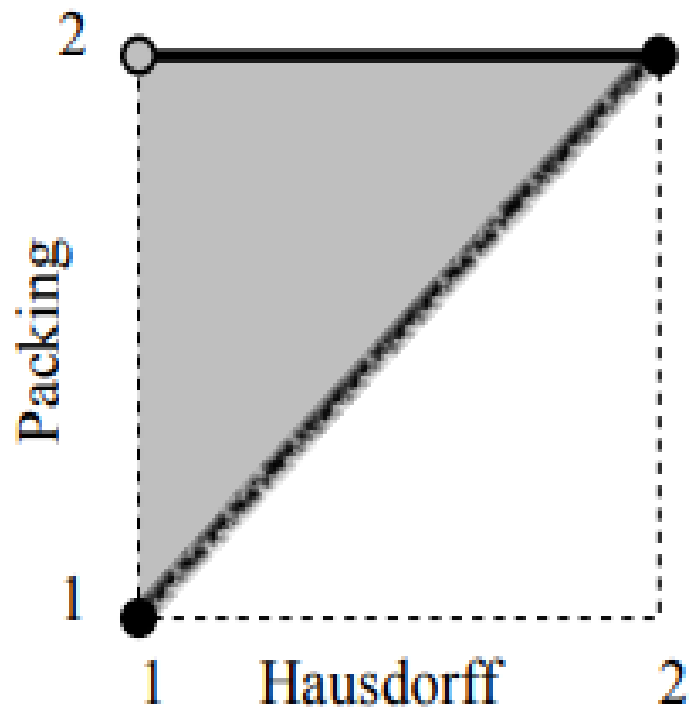


Previous chart of attained dimensions.

Theorem (B.): There exists transcendental entire functions with packing dimension in $(1, 2)$. The set of values attained is dense in $(1, 2)$. Moreover, the packing dimension and Hausdorff dimension may be chosen to be arbitrarily close together (not necessarily equal)..



Updated possible dimensions chart.

Theorem (B.): There exists transcendental entire functions with packing dimension in $(1, 2)$. The set of values attained is dense in $(1, 2)$. Moreover, the packing dimension and Hausdorff dimension may be chosen to be arbitrarily close together.

The Julia set looks like the following:

1. **A fractal quasicircle - the boundary of an attracting basin**
2. Boundaries of Fatou components are C^1 curves - **some not circular**!
3. "Buried" points - **the dimension of the set lives here**!